RIGID MOTIONS, REFLECTIONS AND GROUPS

Rita Desfitri [1]

[1]*Department of Mathematics, University of Bung Hatta, Padang 25133, Indonesia*
desfilwa@yahoo.com

Abstract

Rigid motion is a transformation consisting of some rotations and translations operation which leave a given shape or arrangement unchanged. In other words, a rigid motion of a shape is a way of moving the shape without bending, tearing or distorting it, so that it looks the same. Reflection, on the other hand, is the operation of exchanging all points of a mathematical object with their mirror images (i.e., reflections in a mirror).

This paper is aimed to discuss the set of rigid motion and reflection of some shapes, together with the operation of compositions which form a group, called the group of symmetries, of the shape. The operation of compositions is commonly written in usual order, (for example, if ***r*** means rotation, and ***h*** reflects about horizontal axis-*h*, then the operation "***r o h***" means we do rigid motion ***h*** first, followed by ***r***). This paper also shows that the rigid motion can be written as permutation, but not all permutations are rigid motion.

Keywords: rigid motion.

1. Rigid motions

Weisstein (1999) stated that a rigid motion is a transformation consisting of rotations and translations which leaves a given arrangement unchanged. Similarly, Chick (1998) said that the group of rigid motion (without reflection) of a shape consists of all the ways in which that shape can be placed in the same look, together with the operation of composition.

2. The group of symmetries

The group of all symmetries of a shape consists of all rigid motions plus reflections as well such that the look of the shape remains unchanged. Rigid motion can be discussed in a very large topics, in two or three dimension materials. In physics, to described the motion of the rigid body, they used to system of coordinates, a spaced-fixed system ***X, Y, Z*** to let the body move rigidly, and the moving system ***X, Y, Z*** which is rigidly fixed in the body/material and at the same time, participates in its motions. For this paper, we only discuss about the fixed system where the body moved rigidly. Together with reflection, then the group of symmetries of some regular polygons will be briefly shown.

3. Permutation Group

A permutation of a set X is a bijection from X to itself or in high school mathematics, a permutation of a set X means a rearrangement of its elements. The family of all permutations of a set X, denoted usually by S_X , and is called the symmetric group on X (Rotman, 1996). When $X = \{1, 2, 3, \ldots, n\}$, then S_X is denoted by S_n , and it is called the group of permutation of n elements or the symmetric group on n elements. We know that $|S_n| = n\,!$. Recall that S_2 is an abelian so S_2 is isomorphic to Z_2. However, S_n is non-abelian for $n \geq 3$.

Note that the rigid motion can also be written as permutations. Example,

$$r = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,4\,1\,2 \end{pmatrix} \quad v = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,1\,4\,3 \end{pmatrix} \quad h = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,3\,2\,1 \end{pmatrix} \text{ and } i = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,2\,3\,4 \end{pmatrix}$$

It is, however, easily to see that not all permutations are rigid motion, such as

$$p = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,3\,2\,4 \end{pmatrix} \text{ and } s = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,2\,3\,1 \end{pmatrix}$$

In these cases, p and s are permutation, but they are not rigid motions, since the arrangement of the parallelogram has been changed.

Example: Rigid Motion of a Parallelogram with a right angle.

Consider the group of rigid motion and also the group of symmetries of a rectangle (a parallelogram with a right angle, and consequently, four right angles). We want to determine, with reason, the group which arises from the group of that planar shape.

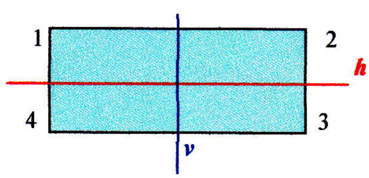

Fig.1 *The simmetries group of Parallelogram*

The element of the group of symmetries would be $P = \{i, r, v, h\}$ where:

- i = identity "motion"
- r = rotate anti-clockwise by Π
- v = reflect about vertical axis – v
- h = reflect about horizontal axis – h

so, the operations of composition can be written as on the following table:

Table 1. The Cayley Table of the group of symmetries of a rectangle

o	*i*	*r*	*v*	*h*
i	*i*	*r*	*v*	*h*
r	*r*	*i*	*h*	*v*
v	*v*	*h*	*i*	*r*
h	*h*	*v*	*r*	*i*

The Cayley table above shows the results of the operations of composition of rigid motions, such as $v \circ r = h = r \circ v$, in other words, rotate anti-clockwise by Π, followed by reflect about vertical axis – v gives result as same as reflect about horizontal axis – h followed by rotate anti-clockwise by Π. This is a four element group, which is, in fact, isomorphic to a direct sum of $Z_2 \oplus Z_2$

4. Dihedrals Group

For each $n \in Z^+$, $n \geq 3$. Let D_{2n} be the set of symmetries of a regular n – gon (Dummit, 23) where a symmetry is any rigid motion of the n – gon which can be affected by taking a copy of the n – gon, moving this copy in any fashion in 3-space an then taking the copy back on the original n – gon so it exactly covers it. D_{2n} is called the dihedral group of **order** $2n$. In some texts the group is written as D_n where the subscript "n" refers to degree or a number of vertices; however, the subscripts give the order of the group rather than a degree is more common in the group theory literate (Dummit, 24).

Let X be any nonempty set and let S_X be the set of all bijection from X to it self (i.e., the set of all permutations of X). The set S_X is called the symmetric group on the set X. In special cases when $X = \{1, 2, 3, \ldots, n\}$, the symmetric group on X and is denoted by S_n, ***the symmetric group of degree n***. It is important to recognize that the elements of S_X are the permutations of X, not the element of X itself. Since the structure of S_X depends only on the cardinality of X, not on the particular element of X itself, it gives result as if X is any finite set with n elements, then the symmetric group S_X "***looks like***" S_n, where S_n is a permutation group of n elements.

5. Rigid Motion of Regular Polygons

We shall see an important family of examples of symmetry groups whose elements a symmetries of geometric objects. For this reasons, we see the simplest subclass of geometric objects, those objects in form of regular planar figures.

a. Triangle

Triangle is a polygon of three sides. A regular triangle is a triangle which is both equilateral (all side of equal length) and equiangular (all interior angles of equal measure).

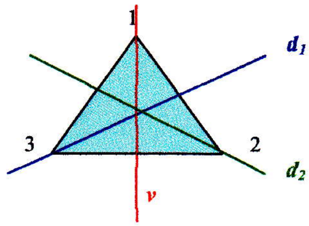

Fig 2. *The symmetries group of Regular Triangle*

The group of symmetries for the regular triangle above would have six elements. Its elements can be listed as in T = $\{i, r_1, r_2, v, d_1, d_2\}$ where :

i = identity motion
r_1 = rotate anti-clockwise by ⅔ Π
r_2 = rotate anti-clockwise by 1 ⅓Π
v = reflect about vertical axis – v
d_1 = reflection about $y = x \tan 30^0 = \frac{1}{3}\sqrt{3}\, x$
d_2 = reflection about $y = x \tan 150^0 = -\frac{1}{3}\sqrt{3}\, x$

Its operations of composition can be written on the following table:

Table 2. The Cayley Table of the group of symmetries of a triangle

o	i	r_1	r_2	v	d_1	d_2
i	i	r_1	r_2	v	d_1	d_2
r_1	r_1	r_2	i	d_1	d_2	v
r_2	r_2	i	r_1	d_2	v	d_1
v	v	d_2	d_1	i	r_2	r_1
d_1	d_1	v	d_2	r_1	i	r_2
d_2	d_2	d_1	v	r_2	r_1	i

By looking at the composition above, it can be seen that the group of symmetries T above is not commutative ($d_1 \circ v = r_1$ but $v \circ d_1 = r_2$), so it can not be isomorphic to a direct sum of $Z_3 \oplus Z_2$ nor Z_6 since none of its elements has an order of six. But then, since a regular triangle is a regular polygon (n – gon) where n equals 3, then the regular triangle can be called the dihedral group of degree 3 and is denoted by D_6. It implies the Cayley table above is, in fact, also isomorphic to D_6. However, since rigid motion (together with reflection) can be written as permutation, that it is isomorphic to S_3. It means that $T = \{i, r_1, r_2, v, d_1, d_2\} \cong D_6 \cong S_3$.

b. Square

Since a square is a regular polygon (n – gon) where n equals 4, then the square can be called the dihedral group of degree 4 and of order 8 so it is denoted by D_8. We shall discuss the rigid motion together with reflection of square, with the operation of their composition which form the group of symmetries.

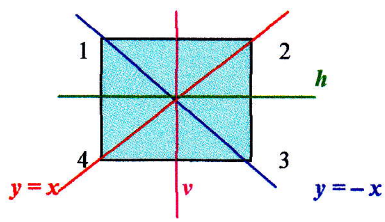

Fig 3. *The symmetries group of Square*

The group of symmetries for the square above would have eight elements. Its elements can be listed as in a group $S = \{i, r_1, r_2, r_3, v, h, d_1, d_2\}$ where :

i = identity motion
r_1 = rotate anti-clockwise by ½ Π
r_2 = rotate anti-clockwise by Π
r_3 = rotate anti-clockwise by 1 ½ Π
h = reflection about horizontal axis – h
v = reflection about vertical axis – v
d_1 = reflection about $y = x$
d_2 = reflection about $y = -x$

and the operations of composition can be written on the Table 3 below.

It can be noted that S is a non-abelian group ($r_1 \circ h = d_1$ and $h \circ r_1 = d_2$). The operation of composition of rotations give rotations, the operations of flips followed by rotations or rotations followed by flips will leave the result as flip operations, but the operation of compositions of the flips (flips followed by flips) give results as rotations.

Table 3. The Cayley Table of the group of symmetries of a square

O	i	r_1	r_2	r_3	h	v	d_1	d_2
I	i	r_1	r_2	r_3	h	v	d_1	d_2
R_1	r_1	r_2	r_3	i	d_1	d_2	h	v
R_2	r_2	r_3	i	r_1	v	h	d_1	d_2
R_3	r_3	i	r_1	r_2	d_2	d_1	h	v
h	h	d_2	v	d_1	i	r_2	r_3	r_1
v	v	d_1	h	d_2	r_2	i	r_1	r_3
D_1	d_1	h	d_2	v	r_1	r_3	i	r_2
D_2	d_2	v	d_1	h	r_3	r_1	r_2	i

By looking at the Cayley Table above, then rotations forms a subgroup of S which is isomorphic to Z_4.

Also note that D_8 can be generated by only two operations. Let g = rotation anti-clockwise by ½ Π and f = reflection through vertical axis (= v). Then the elements of group can be generated from g and f, where

$$i = g^0 = f^0 = g \circ g \circ g \circ g = g^4 = f \circ f = f^2$$
$$r_1 = g, \quad r_2 = g^2, \quad r_3 = g^3,$$
$$v = f, \quad h = g^2 f,$$
$$d_1 = g^3 f = fg, \quad \text{and} \quad d_2 = g f = fg^3,$$

In general cases, if it is given a regular planar polygon with n vertices (n – gon), g represents rotation anti-clockwise by $2\Pi/n$, and f represents reflection about vertical axis. Note that $o(g) = n$, and $o(f) = 2$, (ie. $g^n = i$, and $f^2 = i$). It can be seen that

$$fg = g^{-1} f = g^{n-1} f$$

which eventually give:

$$\begin{aligned}
fg^k &= fg(g^{k-1}) \\
&= g^{n-1} f(g^{k-1}) \\
&= g^{n-1} fg(g^{k-2}) \\
&= g^{n-1}(g^{n-1} f)(g^{k-2}) \\
&= g^{2n-2}(f g^{k-2}) \\
&= g^n g^{n-2}(f g^{k-2}) \\
&= g^{n-2}(f g^{k-2}) \\
&= g^{n-2}(g^{n-(k-2)} f) \\
&= g^{n-2}(g^{n-k+2} f) \\
&= g^{n-2+2}(g^{n-k} f) \\
&= g^{n-k} f
\end{aligned}$$

This implies $fg^k = g^{n-k} f$

Example:

In D_8, we have operation of $fg^2f^3g^5 = fg^2fg$ (since $f^2 = i = g^4$)
So,

$$
\begin{aligned}
fg^2f^3g^5 &= fg^2fg \\
&= fg^2(g^3f) \\
&= fg\,(g^4f) \\
&= (g^3f)(f) \\
&= g^3f^2 \\
&= g^3
\end{aligned}
$$

6. Generalizations/Summaries

For the group of rigid motion of a regular n-gon, (dihedral group D_n), then the following apply:

- $|D_{2n}| = 2n$, the elements of a group of symmetries of a regular n-gon are $n + n = 2n$ which consists of n elements of rotation (by $2\Pi/i$, for $i = 1, \ldots, n$) and also n elements of reflections.

- D_{2n} is non-abelian for $n \geq 3$

- The elements of D_{2n} can also be written in the form of $D_{2n} = <g, f>$ where $D_{2n} = <g, f> = \{i, g, g^2, \ldots, g^{n-1}, f, gf, g^2f, \ldots, g^{n-1}f\}$.

- $<g> = \{i, g, g^2, \ldots, g^{n-1}, g^n = i$), so $<g> \cong Z_n$.

- $<g>$ is a **normal subgroup** of D_{2n} since $(g^k)^{-1}g^rg^k = g^{-k+r+k} = g^r \in <g>$ and for arbitrary element, we have $(g^kf)^{-1}g^rg^kf = f^{-1}g^{-k}g^rg^kf = f^{-1}g^rf = = f^{-1}(fg^{n-r}) = g^{n-r} \in <g>$

- $<f> = \{i, f^2 = i\}$, so $<f> \cong Z_2$.

- $<f>$ is **not a normal subgroup** of D_{2n}, since $(g^r)^{-1}fg^r = g^{-r}fg^r = g^{-r}g^{n-r}f = g^{n-2r}f$. For $n \neq 2r$, then $g^{n-2r}f \notin <f>$
  Spesially, $g^{-1}fg = g^{-1}g^{n-1}f = g^{n-2}f$. For $n \neq 2$, then $g^{n-2}f \notin <f>$

- D_{2n} is (isomorphic to) a subgroup of S_n, since a rigid motion can be written as a permutation. Further, D_6 is (isomorphic to) a subgroup of S_3. D_8 is (isomorphic to) a subgroup of S_4. In special case for $n = 3$, since $|D_{2n}| = 2n$, then $|D_6| = 6$, and of course, $|S_3| = 6$, so $D_6 \cong S_3$.

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